PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, FEBRUARY 13, 2024, ON The LIMIT COMPARISON TEST, MORE ON ALTERNATING SERIES and Sec 11.6 - Absolute CONVERGENCE VS. CONDITIONAL CONVERGENCE, and TESTS FOR ABSOLUTE CONVERGENCE: The Ratio Test and The Root TEST

CLASS #9

Last Time, we saw the (Direct) Comparison Test

Now, we have the Limit Comparison Test.

THE LIMIT Comparison Test

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = c$ AND

c is a finite non-zero number,

then either both series converge

OR

both series diverge.

If $c = 0$ or $c = \infty$, then the test fails and some other test must be used to identify convergence or divergence.

THE LAST SENTENCE ABOVE SHOWS HOW THE LIMIT COMPARISON TEST CAN FAIL.

THE EXAMPLE OF USING THE LIMIT COMPARISON TEST PRESENTED IN CLASS.

EXAMPLE: Is $\displaystyle\sum_{n=1}^{\infty} \frac{4n+3}{\sqrt{n^5+n^3+1}} = \sum_{n=1}^{\infty} a_n$ C or D?

Sol'n: For the comparison series $\displaystyle\sum_{n=1}^{\infty} b_n$, what should b_n be?

$$b_n = \frac{n}{\sqrt{n^5}} = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

THE FIRST JUSIFICATION (REQUIRED)

"$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent."

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{4n+3}{\sqrt{n^5+n^3+1}}\right)\left(\frac{n^{3/2}}{1}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n\,(4+3/n)}{\sqrt{n^5}\,\sqrt{1+1/n^2+1/n^5}}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \underbrace{\frac{n^{5/2}}{n^{5/2}}}_{\text{This equals } 1.} \left(\frac{(4+3/n)}{\sqrt{1+\frac{1}{n^2}+\frac{1}{n^5}}}\right) = 4.$$

THE SECOND JUSTIFICATION (REQ'D).

"Because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent,

and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{4n+3}{\sqrt{n^5+n^3+1}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = 4$ and 4 is a finite non-zero number, the series $\displaystyle\sum_{n=1}^{\infty} \frac{4n+3}{\sqrt{n^5+n^3+1}}$ is convergent by the Limit Comparison Test."

Recall from last time, ALTERNATING SERIES

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ where } b_n = |a_n|$$

OR

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n} b_n \text{ where } b_n = |a_n|$$

The Alternating Series Test:

For $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$,

if (1) $b_{n+1} < b_n$ for all $n \geq 1$

and (2) $\lim_{n\to\infty} b_n = 0$,

Then $\sum_{n=1}^{\infty} a_n$ is Convergent.

FACT: When $\sum_{n=1}^{\infty} a_n$ is a Convergent Alternating Series with Summation $S = \sum_{n=1}^{\infty} a_n$ (and with $b_n = |a_n|$), then, for any integer $n \geq 1$, with the ERROR in $S_n \approx S$ is $|S - S_n|$,

then $|S - S_n| \leq b_{n+1} = |a_{n+1}|$.

So, for $S_n \approx S$, the Level of ERROR is $|a_{n+1}| = b_{n+1}$.

Problem: Let $s = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Find the level of ERROR in using $S_{19} \approx S$.

Sol'n: $S_{19} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots \cdots - \frac{1}{18} + \frac{1}{19}$

$$a_{19} = (-1)^{20} \frac{1}{19} = + \frac{1}{19}.$$

$$a_{20} = (-1)^{21} \frac{1}{20} = - \frac{1}{20} = -0.05..$$

The Level of ERROR in using $S_{19} \approx S$ is $|a_{20}| = \left| -\frac{1}{20} \right| = \frac{1}{20} = 0.05$

Ex: It can be shown that

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is Convergent by the ALT. SERIES TEST.

Let s = the sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$.

What is the Level of Error in using $S_9 \approx S$?

Sol'n

The Level of Error is $S_9 \approx S$ is

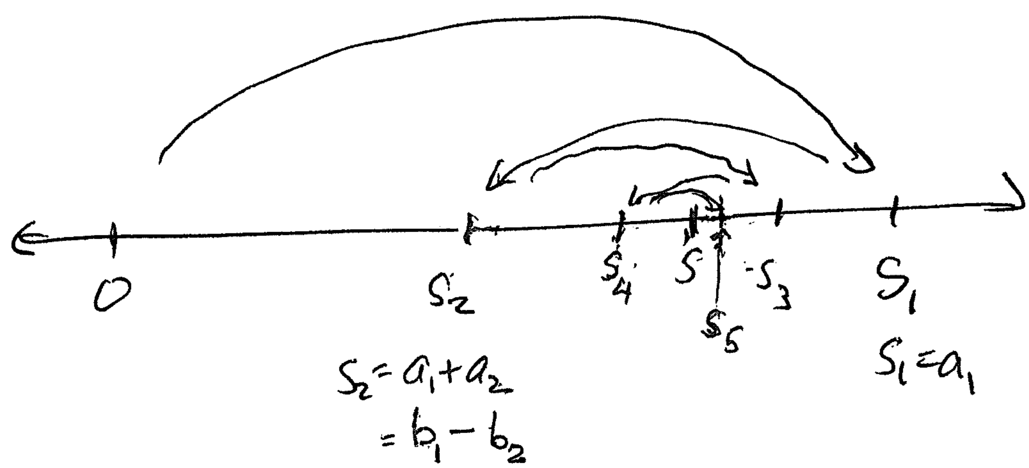
$$|a_{10}| = \left| (-1)^{11} \frac{10^2}{10^3+1} \right| = \frac{100}{1{,}001} \approx \underline{0.00999001}$$

Suppose $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is a

convergent alternating with $\lim_{n\to\infty} S_n = S$ = the sum.

Then, the partial sums S_n approach S

as follows:



How the Partial Sums S_n approach the sum S for a

CONVERGENT ALTERNATING SERIES

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n \quad , \quad b_n = |a_n|$$

Correct to 3 Decimal Places ?????

The Phrase

"The number A approximates sum s correct to k decimal places"

means

"The Error in the approximation,

Error = | s – A | ⩽ 0. 00...05 "

where there are k zeros between '.' and '5'.

Thus, " A approximates sum s correct to 3 places"

means " Error ⩽ 0.0005 ".

Thus, " A approximates sum s correct to 6 places"

means " Error ⩽ 0.0000005 ".

This really says, for k decimal places ,

$$\text{" Error } \leqslant 0.5 \times 10^{-k} \text{ "},$$

and this has k zeros before the 5 since

$$0.5 \times 10^{-k} = 5.0 \times 10^{-(k+1)} .$$

IMPORTANT LIMITS

$\lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$ since $\left(\frac{n+1}{n}\right) = \left(1+\frac{1}{n}\right) \to 1+0$

$\lim_{n\to\infty} \left(\frac{n}{n+1}\right) = 1$ since $\left(\frac{n}{n+1}\right) = 1/\left(\frac{n+1}{n}\right) = 1/\left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ since $\left(1+\frac{1}{n}\right)^n = e^{n\ln(1+\frac{1}{n})}$

$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = 1$
by L'HOSPITAL'S Rule

IN THE FOLLOWING, k is a positive constant.
AS A RESULT, $\ln(k)$ IS constant also.

$\lim_{n\to\infty} \sqrt[n]{k} = 1$ since $\sqrt[n]{k} = k^{\frac{1}{n}} = e^{\frac{1}{n}\ln(k)} \to e^0 = 1$

$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty}\left(n^{\frac{1}{n}}\right) = 1$ since $n^{\left(\frac{1}{n}\right)} = e^{\left(\frac{1}{n}\ln(n)\right)}$

and by L'HOSPITAL'S RULE,
$\lim_{x\to\infty} \frac{1}{x}\ln(x) = 0$

$\lim_{n\to\infty} \sqrt[n]{n^k} = 1$
since
$\sqrt[n]{n^k} = \left(n^k\right)^{\frac{1}{n}} = \left(n^{\frac{1}{n}}\right)^k = \sqrt[n]{n}^{\,k} \to 1^k = 1$

Def'n: Given a series $\sum_{n=1}^{\infty} a_n$, its Corresponding ABSOLUTE SERIES is $\sum_{n=1}^{\infty} |a_n|$.

Series	Corresponding Absolute Series
IF → $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is C	$\sum_{n=1}^{\infty} \frac{1}{n}$ is D
$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is C	$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is C ← IF

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is Absolutely Convergent ← Then

THEN → $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is Conditionally Convergent.

In General,

the Alternating Series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$

is exactly one of these:

(1) Absolutely Convergent

(2) Conditionally Convergent

(3) Divergent.

<u>FACT</u>: If an alternating Series is Absolutely Convergent, then the Alternating Series is Convergent.

Absolute Convergence implies Convergence.

TWO ROOT TEST EXAMPLES AND TWO RATIO TEST EXAMPLES

The TESTS FOR ABSOLUTE CONVERGENCE

Let $\sum_{n=1}^{\infty} a_n$ be a given series.

Consider the following limit L.
(Both tests interpret L in the SAME WAY.)

THE RATIO TEST	THE ROOT TEST
$\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| = L$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L$

If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is Absolutely Convergent.

If $L > 1$, Then $\sum_{n=1}^{\infty} a_n$ is Divergent.

If $L = 1$, the test fails and some other test must be used.

In what follows,
in EXAMPLE 1, the RATIO Test is applied
In EXAMPLE 2, the ROOT Test is applied.

EXAMPLE 1 :

Is the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ Convergent OR Divergent?

Solution (USING the RATIO TEST):

$$|a_n| = \frac{n^3}{3^n} \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)^3}{3^{(n+1)}}$$

$$L = \lim_{n\to\infty} \left(\frac{\frac{(n+1)^3}{3^{(n+1)}}}{\frac{n^3}{3^n}} \right) = \lim_{n\to\infty} \left(\frac{(n+1)^3}{3^{(n+1)}}\right)\left(\frac{3^n}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{3}\right)\left(\frac{n+1}{n}\right)^3 = \left(\frac{1}{3}\right)(1^3) = \frac{1}{3} = L$$

since $\lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$.

JUSTIFICATION IS REQUIRED

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$ and $\frac{1}{3} < 1$,

the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is Absolutely Convergent by the RATIO Test.

[Of course, by an earlier theorem, since $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent, it is convergent.]

Problem : Is the Series $\sum_{n=1}^{\infty} \frac{3^n}{n^5}$ Convergent or Divergent?

Solution (Using the Root test)

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^5}} = \lim_{n\to\infty} \frac{\sqrt[n]{3^n}}{\sqrt[n]{n^5}}$$

$$= \lim_{n\to\infty} \frac{3}{\sqrt[n]{n^5}} = \frac{\lim_{n\to\infty} 3}{\lim_{n\to\infty} \sqrt[n]{n^5}} = \frac{3}{1} = 3 = L,$$

Since $\lim_{n\to\infty} \sqrt[n]{n^5} = \lim_{n\to\infty} (\sqrt[n]{n})^5 = 1$

Since $\lim_{n\to\infty} \sqrt[n]{3^n/n^5} = 3$ and $3 > 1$,

the Series $\sum_{n=1}^{\infty} \frac{3^n}{n^5}$ is Divergent by the Root Test.